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OSCILLATIONS IN SYSTEMS WITH NON-LINEAR DAMPING†

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The conditions for the existence of not less than two cycles in a two-dimensional dynamical system with a single scalar non-linearity are obtained. The results are compared with those of Keldysh [1, 2].

CONSIDER the second-order differential equation

$$x^{\cdot\cdot}+\varphi(x^{\cdot})+x=0 \tag{1}$$

where $\varphi(\sigma)$ is an odd function which is differentiable when $\sigma\neq 0$ and which has a discontinuity of the first kind when $\sigma=0$. We will refer to the solution of Eq. (1) in the sense of Filippov [3]. Henceforth, we shall consider the phase space of the system $x^{\cdot}=y,\ y^{\cdot}=-\varphi(y)-x$ as the phase space of Eq. (1).

Equation (1) and its various generalizations have been studied in a number of publications [4, 5]. The role of Eq. (1) in eliminating flutter of control devices of an aircraft using hydraulic dampers was demonstrated in [1, 2].

Let us assume that the lower limit as $\sigma\to\infty$ of the function $\varphi(\sigma)/\sigma$ is greater than a certain positive number and that the inequalities $\beta>2$ and

$$-\alpha\sigma+\varphi(+0)\leqslant\varphi(\sigma)\leqslant-\beta\sigma+\varphi(+0),\quad \forall\sigma\in[0,\gamma] \tag{2}$$

are satisfied in the case of the positive numbers α, β and γ.

We introduce the notation $\xi=\pi\beta/\sqrt{(4-\beta^2)}$.

Theorem. Let

$$2\varphi(+0)e^{\xi}/(e^{\xi}-1)<\gamma \tag{3}$$

Then Eq. (1) has not less than two cycles.

Proof. The existence of positive numbers λ and ν, for which the inequality

$$\varphi(\sigma)\geqslant\lambda\sigma-\nu,\quad \forall\sigma\in[0,+\infty) \tag{4}$$

holds, follows from the condition for the behaviour of the function $\varphi(\sigma)$ at infinity.

Using the Kamke–Chaplygin comparison principle [6, 7], it is seen that the cycles of the equations

$$z^{\cdot\cdot}+\lambda z^{\cdot}-\nu\,\mathrm{sign}z^{\cdot}+z=0 \tag{5}$$

$$z^{\cdot\cdot}-\alpha z^{\cdot}+\varphi(+0)\mathrm{sign}z^{\cdot}+z=0 \tag{6}$$

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$$z^{\cdot\cdot} - \beta z^{\cdot} + \varphi(+0)\mathrm{sign} z^{\cdot} + z = 0 \tag{7}$$

are contactless in the case of the trajectories of Eq. (1). This follows from inequalities (2) and (4) and the fact that, by virtue of condition (3), the cycle (7) is located in a band $|z^{\cdot}| \leq \gamma$.

Hence, the ring K_1, with boundaries which are the cycles of Eqs (5) and (7), is positively invariant in the case of the trajectories of Eq. (1) while the ring K_2, the boundaries of which are the cycles of Eqs (6) and (7), is negatively invariant in the case of the trajectories of Eq. (1). Then, using the well-known ring principle [8], we next complete the proof of the theorem.

Keldysh considered Eq. (1) in the case when

$$\varphi(\sigma) = -\mu\sigma + (\Phi + \kappa\sigma^2)\mathrm{sign}\sigma$$

In this case, inequality (3) takes the form

$$2\Phi e^{\xi}/(e^{\xi} - 1) < (\mu - \beta)/\kappa \tag{8}$$

When $\beta \ll 1$, putting $\beta = \mu/2$, we can write condition (3) of the theorem in the case of small μ in the form $\mu > 4\sqrt{(\Phi\kappa/\pi)}$. This estimate is close to the estimate $\mu > (8/\pi)\sqrt{(2\Phi\kappa/3)}$ obtained in [1, 2], using the approximate method of harmonic balance.

Note that the apparatus of reference systems for multidimensional dynamical systems which has recently been developed [8,9] enables one to make certain multidimensional generalizations of the theorem which has been presented here.

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